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Rank rigidity for foliations by manifolds of nonpositive curvature

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Abstract: We develop a rank rigidity theorem for finite volume foliations by manifolds of nonpositive sectional curvature. It implies that if the leaves are irreducible Hadamard manifolds of geometric rank $\geqslant 2$, then the leaves are symmetric.

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Introduction

We wish to generalize a result [9, Corollary 1, p. 33] of P. Eberlein and J. Heber, thereby settling a question posed by S. Hurder at the problem session on rigidity and geodeic flows held on 30 May, 1984 at MSRI [5, Problem 2.5, p. 309].

We will prove:

Theorem 8.4. *Let $\mathcal{F}$ be a measure preserving foliation of a finite measure space by connected, complete Riemannian manifolds. Assume that a.e. leaf has nonpositive sectional curvature. Then, for a.e. leaf L, all of the nonsymmetric irreducible factors of the universal cover of L have geometric rank $= 1$.*

For the definition of geometric rank, see Definition 4.1. For general information about manifolds of nonpositive sectional curvature, see [4].

For the definition of a measure preserving foliation of a finite measure space by Riemannian manfolds, see [1, Definition 1.5], where the terminology "Riemannian foliated measure space with finite total volume" is used instead. Much study of ergodic theoretic foliations (i.e., foliations with only a measure-theoretic structure in the transverse direction) has been done by R. Zimmer. In particular [11, Corollary 3.5, p. 48] shows

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that, up to finite-to-one covers, the problem of classifying ergodic foliations by Riemannian globally symmetric spaces of noncompact type is equivalent to the problem of understanding free, finite measure preserving, ergodic actions of the corresponding semisimple Lie groups. With regard to this latter problem, much is known, see [12], especially [12, Theorem 5.2.1, p. 95] and [12, Theorem 5.2.5, p. 98].

Unfortunately, we know of no connected, simply connected complete Riemannian manifold of nonpositive curvature which is irreducible, has geometric rank $\geqslant 2$ and is nonsymmetric. (In other words, we do not know if the assumption that the isometry group satisfy the duality condition is really necessary in [9, Theorem 4.1, p. 41].) In the unlikely event that such manifolds do not exist, then Theorem 8.4 becomes true for trivial reasons.

Let $\mathcal{F}$ be a C^∞-foliation of a Riemannian manifold. Then each leaf is a Riemannian manifold with the inherited Riemannian structure. Assume that there is a transverse invariant measure such that the integral of the leafwise volume against the transverse measure has finite total volume. Then $\mathcal{F}$ is an example of a measure preserving foliation of a finite measure space with Riemannian leaves. Therefore, if we also assume that the leaves are Hadamard manifolds whose irreducible factors are all of geometric rank $\geqslant 2$, then Theorem 8.4 implies that the a.e. leaf is symmetric. In this case, since all derivatives of the metric tensor are transverse-continuous, the a.e. vanishing of the covariant derivative ∇R of leafwise curvature implies that ∇R is identically zero, i.e., that *every* leaf is symmetric.

By considering one leaf foliations, one sees that Theorem 8.4 implies [9, Corollary 1, p. 33]. In the case of a single leaf with sectional curvature bounded below, rank rigidity was originally proved by W. Ballman [2] and by K. Burns and R. Spatzier [7], see also [6].

Let $\tilde{M}_1$, $\tilde{M}_2$ be Hadamard manifolds. Assume that $\tilde{M}_1$ is a product of irreducible manifolds of geometric rank $= 1$ and that $\tilde{M}_2$ is a product of irreducible manifolds of geometric rank $\geqslant 2$. Assume that $\tilde{M}_1 \times \tilde{M}_2$ admits a finite volume quotient M. By the one leaf case of Theorem 8.4, $\tilde{M}_2$ is symmetric. (Alternatively, M is foliated by quotients of $\tilde{M}_2$ and Theorem 8.4 again implies that $\tilde{M}_2$ is symmetric.) This result may also be obtained from [9, Theorem 4.1, p. 41], since it is possible to prove that the isometry group of $\tilde{M}_2$ satisfies the duality condition.

The general line of argument for proving Theorem 8.4 is about the same as that in [9, Theorem 4.1, p. 41]. However, the usages of the duality condition must be replaced by measure-theoretic recurrence of the geodesic flow. This can be troublesome, since both of the two formulations of the duality condition (see [8, Remark 2, p. 741]) are topological (not measure-theoretic) in nature. The second definition in [8, p. 741] is particularly difficult since it is not at first glance related to any sort of recurrence of the geodesic flow.

In Section 1, we introduce some terminology and basic facts about smooth manifolds. In Section 2, we place basic results about manifolds of nonpositive curvature. In Section 3, we show how the existence of a certain kind of vectors ("reverse recurrent vectors") implies the existence of geodesically convex submanifolds that in Section 4 are shown to be flat. In Section 4, we study Hadamard manifolds with "enough flats". In Section 5, we show that, for manifolds with enough flats, the existence of a certain kind of vectors ("forward recurrent vectors") implies a certain behavior (a flat covers a Tits ball at infinity) which in Section 6 is shown to imply that the metric-space geometry of some Tits ball is, in some sense, visible from all points in the Hadamard manifold. In Section 6, we use a result of P. Eberlein to show that this "Tits-visibility" combined with the existence of "horizon sets" at infinity together imply that the manifold is symmetric. In Section 7, we split the universal cover of a nonpositively curved manifold into its deRham factors, and study the extent to which the main results of Section 3–6 split along with it. None of the results in Section 2–7 is related to foliations. In Section 8, we consider finite volume foliations and show that Poincaré recurrence guarantees the existence of many reverse recurrent and forward recurrent vectors. We use another ergodic theoretic result to produce horizon sets on the higher rank factors of the universal cover of a generic leaf. The machinery of Section 7 then shows that such factors are in fact symmetric.

The results in Section 2–6 represent a modularization and reformulation of similar results from [3] and [9]. (We have attempted to indicate this at the relevant points by referring to the earlier articles, and apologize if there has been any omission.) This reformulation was necessary for our purposes, since the propositions we need have not before been stated in exactly the form we require. In particular, none of the results stated in Section 2–6 (or in the rest of this paper, for that matter) require the duality condition.

I owe R. Zimmer a debt of gratitude for suggesting this line of research. He declined to coauthor the paper but deserves significant credit in the development of this theorem. I would also like to thank P. Eberlein for a number of lucid and very helpful conversations.

1. Generalities on smooth manifolds

Throughout this section, all manifolds are assumed to be C^∞-smooth manifolds without boundary. None of the results in Section 1 depend on a Riemannian structure.

We will say that a manifold is *Euclidean* if it is diffeomorphic to some Euclidean space $\mathbb{R}^d$.

Definition 1.1. Let A be a manifold, let $U \subseteq A$ be open, let $S \subseteq A$ and let k be a nonnegative integer. Let $d := \dim(A)$. We say that S is *k-smooth on U* if there exists a diffeomorphism $f : \mathbb{R}^d \to U$ such that $f(\mathbb{R}^k \times \{0\}^{d-k}) = S \cap U$. We say that S is *locally k-smooth on U* if, for every $u \in U$, there exists an Euclidean neighborhood $U_0 \subseteq U$ of u such that either $(S \cap U_0 = \emptyset)$ or (S is k-smooth on U_0).

If S is k-smooth on U, then S is locally k-smooth on U. If U and U' are both open, if $U \subseteq U'$ and if S is locally k-smooth on U', then S is locally k-smooth on U. If S is locally k-smooth on A, then we say that S is a *k-submanifold of A*.

If A, Z and B are manifolds, and if $H : A \times Z \to B$ is C^∞-smooth, then we define $H_z := a \mapsto H(a,z) : A \to B$, for all $z \in Z$. We also define $\tilde{H} := (a,z) \mapsto (H(a,z),z) : A \times Z \to B \times Z$. If $b_0 \in B$, then we say that H is a *Z-chart about b_0* if A is Euclidean and if there exists a neighborhood B_0 of b_0 such that $\tilde{H}$ is a diffeomorphism of $A \times Z$ onto some open set in $B \times Z$ containing $B_0 \times Z$.

The next two results are parametric implicit function theorems; the first submersive, the second immersive.

Lemma 1.2. *Let B, Z and L be manifolds. Let $G : B \times Z \to L$ be C^∞-smooth. Let $(b_0,z_0) \in B \times Z$. Assume that $G_{z_0} : B \to L$ is submersive at b_0. Then there exist*

(1) *open neighborhoods Z_0 of z_0 and L_0 of $G(b_0,z_0)$;*

(2) *an Euclidean manifold D_0 of dimension* $\dim(B) - \dim(L)$; *and*

(3) *a Z_0-chart $\Phi : D_0 \times L_0 \times Z_0 \to B$ about b_0; such that $(G \circ \Phi)(d,l,z) = l$, for all $(d,l,z) \in D_0 \times L_0 \times Z_0$.*

Proof. Since G_{z_0} is submersive at b_0, it follow that $\tilde{G}$ is submersive at (b_0,z_0) and so the usual implicit fucntion theorem yields an Euclidean manifold D_0, an open neighborhood $U_0 \subseteq L \times Z$ of $\tilde{G}(b_0,z_0)$ and a chart $\bar{\Phi} : D_0 \times U_0 \to B \times Z$ about (b_0,z_0) such that $(\tilde{G} \circ \bar{\Phi})(d,u) = u$, for all $(d,u) \in D_0 \times U_0$. Possibly replacing U_0 by a smaller neighborhood of $\tilde{G}(b_0,z_0)$, we may assume that $U_0 = L_0 \times Z_0^*$, for some neighborhoods L_0 of $G(b_0,z_0)$ and Z_0^* of z_0.

Let Φ denote the composition of $\bar{\Phi}$ followed by the projection map $B \times Z \to B$.

We claim that $\tilde{\Phi} = \bar{\Phi}$. Fix $(d,l,z) \in D_0 \times L_0 \times Z_0^*$. Let $b := \Phi(d,l)$. Then $\tilde{\Phi}(d,l,z) = (b,z)$, while $\bar{\Phi}(d,l,z) = (b,z')$, for some $z' \in Z$; we wish to show that $z' = z$. However,

$$(G(b,z'),z') = \tilde{G}(b,z') = (\tilde{G} \circ \bar{\Phi})(d,l,z) = (l,z),$$

so comparing last coordinates yields $z' = z$, proving the claim.

Then $\tilde{\Phi} : D_0 \times L_0 \times Z_0^* \to B \times Z$ is a chart about (b_0,z_0), so we may choose neighborhoods B_0 of b_0 and $Z_0 \subseteq Z_0^*$ of z_0 such that $B_0 \times Z_0 \subseteq \tilde{\Phi}(D_0 \times L_0 \times Z_0^*)$. It then follows from the definition of $\tilde{\Phi}$ that $B_0 \times Z_0 \subseteq \tilde{\Phi}(D_0 \times L_0 \times Z_0)$.

Since $(\tilde{G} \circ \tilde{\Phi})(d,l,z) = (\tilde{G} \circ \bar{\Phi})(d,l,z) = (l,z)$, it follows, by projecting to first coordinates, that $(G \circ \Phi)(d,l,z) = l$, for all $(d,l,z) \in D_0 \times L_0 \times Z_0$. $\square$

Lemma 1.3. *Let A, Z and L be manifolds. Let $F : A \times Z \to L$ be C^∞-smooth. Let $(a_0,z_0) \in A \times Z$. Assume that $F_{z_0} : A \to L$ is immersive at a_0. Then there exist*

(1) *open neighborhoods A_0 of a_0 and Z_0 of z_0;*

(2) *an Euclidean manifold C_0 of dimension* $\dim(L) - \dim(A)$ *and a point $c_0 \in C_0$; and*

(3) *a Z_0-chart $\Psi : C_0 \times A_0 \times Z_0 \to L$ about $F(a_0,z_0)$; such that $\Psi(c_0,a,z) = F(a,z)$, for all $(a,z) \in A_0 \times Z_0$.*

Proof. By passing to coordinates, we may assume that L is an Euclidean space and that $F(a_0,z_0) = 0$. Let $f := F_{z_0}$. Let C denote Euclidean space of dimension $\dim(L) -$

$\dim(A)$. Let $c_0 := 0$. Now choose a C^∞-smooth map $g : C \to L$ such that $g(0) = 0$ and such that $(dg)(T_0C) + (df)(T_{a_0}A) = T_0L$.

Now apply Lemma 1.2 with $B := C \times A$, $b_0 := (0, a_0)$ and

$$G := (c, a, z) \mapsto g(c) + F(a, z) : B \times Z \to L.$$

In this case, $\dim(B) = \dim(L)$, so D_0 will consist of a single point. We thus obtain a Z_0-chart $\Phi : L_0 \times Z_0 \to C \times A$ about $(0, a_0)$ such that $(G \circ \Phi)(l, z) = l$.

By definition of a Z_0-chart, we may choose neighborhoods C_0 of 0 and A_0 of a_0 such that $C_0 \times A_0 \times Z_0 \subseteq \tilde{\Phi}(L_0 \times Z_0)$. We may also assume that C_0 is an open ball in the Euclidean space C and hence is an Euclidean manifold.

The inverse of $\tilde{\Phi}$ is defined on $C_0 \times A_0 \times Z_0$ and we denote this inverse

$$\bar{\Psi} : C_0 \times A_0 \times Z_0 \to L_0 \times Z_0.$$

Let Ψ be the composite of $\bar{\Psi}$ followed by the projection map $L_0 \times Z_0 \to L_0$.

We claim that $\tilde{\Psi} = \bar{\Psi}$. Fix $(c, a, z) \in C_0 \times A_0 \times Z_0$. Let $l := \Psi(c, a)$. Then $\tilde{\Psi}(c, a, z) = (l, z)$, while $\bar{\Psi}(c, a, z) = (l, z')$, for some $z' \in Z$; we wish to show that $z' = z$. However,

$$(\Phi(l, z'), z') = \tilde{\Phi}(l, z') = (\tilde{\Phi} \circ \bar{\Psi})(c, a, z) = (c, a, z),$$

so comparing last coordinates yields $z' = z$, proving the claim.

Let $(a, z) \in A_0 \times Z_0$; we wish to show that $\Psi(0, a, z) = F(a, z)$, or, equivalently, that $\tilde{\Psi}(0, a, z) = \tilde{F}(a, z)$. Since $\tilde{G} \circ \tilde{\Phi} = \tilde{G} \circ \bar{\Phi}$ is the identity on $L_0 \times Z_0$, it suffices to show that $(\tilde{G} \circ \tilde{\Phi} \circ \tilde{\Psi})(0, a, z) = \tilde{F}(a, z)$. Since $\tilde{\Phi} \circ \tilde{\Psi}$ is the identity on $C_0 \times A_0 \times Z_0$, we are reduced to showing that $\tilde{G}(0, a, z) = \tilde{F}(a, z)$, or, equivalently, that $G(0, a, z) = F(a, z)$. This follows from the definition of G. $\square$

Corollary 1.4. *Let A, Z and L be manifolds and let $F : A \times Z \to L$ be C^∞-smooth. Let $(a_0, z_0) \in A \times Z$ and assume that $F_{z_0} : A \to L$ is immersive at a_0. Then there exist neighborhoods A_0 of a_0, Z_0 of z_0 and L_0 of $F(a_0, z_0)$ such that, for all $z \in Z_0$, $F_z(A_0)$ is locally* $\dim(A)$*-smooth on L_0.*

Proof. Lemma 1.3 yields the neighborhoods A_0 and Z_0, along with an Euclidean manifold C_0, a point $c_0 \in C_0$ and a Z_0-chart $\Psi : C_0 \times A_0 \times Z_0 \to L$ about $F(a_0, z_0)$. Replacing A_0 by a possibly smaller neighborhood of a_0, we may assume that A_0 is Euclidean. By definition of a Z_0-chart, there exists a neighborhood L_0 of $F(a_0, z_0)$ such that $L_0 \times Z_0 \subseteq \tilde{\Psi}(C_0 \times A_0 \times Z_0)$.

Let A_0^* and C_0^* be Euclidean spaces of dimension $\dim(A_0)$ and $\dim(C_0)$ and choose diffeomorphisms $A_0^* \to A_0$ and $C_0^* \to C_0$ such that $0 \mapsto a_0$ and $0 \mapsto c_0$. For $z \in Z_0$, let Ψ_z^* denote the composite of $C_0^* \times A_0^* \to C_0 \times A_0$ followed by Ψ_z. Then, for every $z \in Z_0$, we have $\Psi_z^*(\{0\} \times A_0^*) = F_z(A_0) \cap \Psi_z(A_0 \times C_0)$.

Thus, for every $z \in Z_0$, we have shown that $F_z(A_0)$ is $\dim(A)$-smooth on $\Psi_z(A_0 \times C_0)$. However $L_0 \subseteq \Psi_z(A_0 \times C_0)$, for every $z \in Z_0$, and local smoothness on L_0 follows. $\square$

Lemma 1.5. *Let A be a manifold, let $S \subseteq T \subseteq A$, let k be a nonnegative integer and let $U \subseteq A$ be open and Euclidean. Assume that S is locally k-smooth on U and that T is k-smooth on U. Then $S \cap U = T \cap U$.*

Proof. From Definition 1.1 we conclude that $S \cap U$ is relatively closed in U and is therefore relatively closed in $T \cap U$. From the inverse function theorem, $S \cap U$ is relatively open in $T \cap U$. Finally, by Definition 1.1, $T \cap U$ is connected. □

2. Generalities on Hadamard manifolds

Throughout this section, $\tilde{L}$ denotes a connected, complete, simply connected Riemannian manifold of nonpositive sectional curvature, i.e., a *Hadamard manifold.* None of the results in Section 2 depend on the duality condition or on the existence of a quotient of $\tilde{L}$ of finite volume.

If A is any metric space, $a \in A$ and $r > 0$, then we denote the open ball in A about a of radius r as $B_A(a,r) := \{a' \in A \mid \mathrm{dist}_A(a,a') < r\}$.

We require the *geometric boundary of* $\tilde{L}$, which we denote $\tilde{L}(\infty)$, see [4, Section I.3.2, p. 22]. We use *geodesic* to refer only to unit speed geodesics. If γ is a geodesic, then its asymptotes in $\tilde{L}(\infty)$ are denoted $\gamma(-\infty)$, $\gamma(\infty)$.

Let k be a positive integer and let $F \subseteq \tilde{L}$. Then we say that F is a *k-flat of* $\tilde{L}$ if

(1) F is a k-submanifold of $\tilde{L}$;

(2) F is geodesically convex in $\tilde{L}$; and

(3) with the inherited Riemannian metric, F is isometric to flat Euclidean k-space.

We will use without comment the fact that k-flats are k-smooth on $\tilde{L}$. (*Proof*: Fix a positive integer k and a k-flat F; we wish to show that F is k-smooth on $\tilde{L}$. Fix a point $f \in F$. Then T_fF is k-smooth on $T_f\tilde{L}$ and $\exp_f : T_f\tilde{L} \to \tilde{L}$ is a diffeomorphism, so $\exp_f(T_fF)$ is k-smooth on $\tilde{L}$. As F is complete and geodesically convex, $F = \exp_f(T_fF)$, proving the fact.)

If $F \subseteq \tilde{L}$, then we say that F is a *flat* if F is a k-flat for some positive integer k. Assumption (2) on geodesic convexity is essential: in hyperbolic space, a horosphere in the inherited metric is isometric to flat Euclidean space, but it is not considered to be a flat.

If $A \subseteq \tilde{L}$, then we denote by $A(\infty)$ the intersection of $\tilde{L}(\infty)$ with the closure of A in $\tilde{L} \cup \tilde{L}(\infty)$. If $A \subseteq \tilde{L}$ and $B \subseteq \tilde{L}(\infty)$, then we say that A *covers* B if $B \subseteq A(\infty)$.

Let $T\tilde{L}$ and $S\tilde{L}$ denote the tangent bundle and unit tangent bundle of $\tilde{L}$. The projection map $T\tilde{L} \to \tilde{L}$ is denoted π. The *footpoint* of a vector $v \in T\tilde{L}$ is the point $\pi(v) \in \tilde{L}$.

For every $v \in S\tilde{L}$, let γ_v denote the geodesic such that $\dot{\gamma}_v(0) = v$. The geodesic flow is denoted $\{g_t\}_{t \in \mathbb{R}}$, i.e., for all $v \in S\tilde{L}$, for all $t \in \mathbb{R}$, we define $g_t(v) = \dot{\gamma}_v(t)$.

If $l, l' \in \tilde{L}$ and $l \neq l'$, then we define $\gamma_{ll'}$ to be the geodesic such that $\gamma_{ll'}(0) = l$ and such that, for some $T > 0$, $\gamma_{ll'}(T) = l'$. If $p \in \tilde{L}(\infty)$, $l \in \tilde{L}$, then we define γ_{lp} to be the geodesic such that $\gamma_{lp}(0) = l$ and $\gamma_{lp}(\infty) = p$.

If $p \in \tilde{L}(\infty)$, $l, l' \in \tilde{L}$ and $l \neq l'$, then we define $\angle_l(l', p)$ to be the angle between $\dot{\gamma}_{ll'}(0)$ and $\dot{\gamma}_{lp}(0)$. If $p, q \in \tilde{L}(\infty)$, $l \in \tilde{L}$, then we define $\angle_l(p, q)$ to be the angle between $\dot{\gamma}_{lp}(0)$ and $\dot{\gamma}_{lq}(0)$. For $p, q \in \tilde{L}(\infty)$, we define $\angle(p, q) := \sup_{l \in \tilde{L}} \angle_l(p, q)$.

Remark 2.1. [4, p. 8, l. +6] Let $l, l' \in \tilde{L}$, $x \in \tilde{L}(\infty)$. Let Δ be the geodesic triangle with vertices l, l', x. Let α and α' be the angles of Δ at l and l'. Then $\alpha + \alpha' < \pi$.

Lemma 2.2. *Let $\gamma : \mathbb{R} \to \tilde{L}$ be a geodesic. Fix $q \in \tilde{L}(\infty)$. Then*

$$t \mapsto \angle_{\gamma(t)}(\gamma(\infty), q) : \mathbb{R} \to [0, \pi]$$

is nondecreasing.

Proof. For $t \in \mathbb{R}$, define $\phi(t) := \angle_{\gamma(t)}(\gamma(\infty), q)$. Fix $u, v \in \mathbb{R}$ such that $u < v$; we wish to show that $\phi(u) \leqslant \phi(v)$. Applying Remark 2.1 to the triangle with vertices $\gamma(u)$, $\gamma(v)$ and q, we see that $\phi(u) + (\pi - \phi(v)) \leqslant \pi$. □

Lemma 2.3. *Let $\gamma : \mathbb{R} \to \tilde{L}$ be a geodesic and let $w \in S\tilde{L}$ satisfy $\gamma_w(-\infty) = \gamma(-\infty)$. For every $t \in \mathbb{R}$, let β_t be the geodesic such that $\beta_t(0) = \gamma(t)$ and, for some $T_t > 0$, $\beta_t(T_t) = \pi(w)$. Let $\phi(t)$ denote the angle between $\dot{\gamma}(t)$ and $\dot{\beta}_t(0)$. Let $\psi(t)$ denote the angle between $\dot{\beta}_t(T_t)$ and w. Then $\phi(-t) \to 0$ and $\psi(-t) \to 0$ as $t \to \infty$.*

Proof. Applying Remark 2.1 to the geodesic triangle with vertices $\gamma(t)$, $\pi(w)$ and $\gamma(-\infty)$, we get $(\pi - \phi(t)) + \psi(t) \leqslant \pi$, so $\psi(t) \leqslant \phi(t)$, for all $t \in \mathbb{R}$. So it suffices to show that $\phi(-t) \to 0$ as $t \to \infty$.

Let $\epsilon > 0$; we must show $\phi(-t) < \epsilon$, for sufficiently large t.

Let $d := \mathrm{dist}_{\tilde{L}}(\gamma(0), \pi(w))$. There exists $r > 0$ such that: if Δ is any Euclidean triangle with one side S of length d and another side of length $> r$, then the angle in Δ opposite S is $< \epsilon$.

For $t > r$, we apply the Toponogov comparison theorem [4, p. 6, l. −6 to l. −4] to the geodesic triangle with vertices $\pi(w)$, $\gamma(-t)$ and $\gamma(0)$ and conclude $\phi(-t) < \epsilon$. □

Lemma 2.4. *Let γ_1 and γ_2 be geodesics such that $\gamma_1(-\infty) = \gamma_2(-\infty)$. Then*

$$\angle_{\gamma_1(-t)}(\gamma_1(\infty), \gamma_2(\infty)) \to 0$$

as $t \to \infty$.

Proof. (Suggested by P. Eberlein.) With $z := \gamma_2(-\infty)$, $w := \gamma_2(\infty)$ and $c(t) := \gamma_1(-t)$, this is a consequence of [4, Lemma I.4.2, p. 33]. Note that $\angle(z, w) = \pi$ in this case, since the geodesic γ_2 connects z to w. □

Lemma 2.5. *Let $l \in \tilde{L}$ and let k be a positive integer. Let $F_1, F_2, \ldots$ be a sequence of k-flats such that $d(l, F_i) = 1$, for all i. For each i, choose a point $l_i \in F_i$ and a point $q_i \in F_i(\infty)$. Assume that $l_i \to x \in \tilde{L} \cup \tilde{L}(\infty)$ in the cone topology on $\tilde{L} \cup \tilde{L}(\infty)$.*

Assume that $q_i \to q$ in the cone topology on $\tilde{L}(\infty)$. Then there is a k-flat F such that $x \in F \cup F(\infty)$, $q \in F(\infty)$ and $d(l, F) = 1$.

Proof. For $i = 1, 2, \ldots$, choose $f_i \in F_i$ such that $\mathrm{dist}_{\tilde{L}}(l, f_i) = 1$ and choose an ordered orthonormal basis $\mathcal{B}_i$ for $T_{f_i} F_i$.

After passing to a subsequence, f_i converges to some point $f \in \tilde{L}$. After passing to a further subsequence, $\mathcal{B}_i$ converges to an ordered orthonormal basis for some subspace $V \subseteq T_f \tilde{L}$. Let $F := \exp_f(V)$. □

If $v, w \in S\tilde{L}$, then we say that v and w are *asymptotic* if $\gamma_v(\infty) = \gamma_w(\infty)$. We say that v and w are *parallel* if (v and w are asymptotic) and ($-v$ and $-w$ are asymptotic). For $v \in S\tilde{L}$, the set of footpoints of vectors parallel to v is a closed subset of $\tilde{L}$ which we denote by $\mathrm{FtPar}_{\tilde{L}}(v)$. By the Flat Strip Lemma [10, Proposition 5.1, p. 66], $\mathrm{FtPar}_{\tilde{L}}(v)$ is geodesically convex.

Note that if F is a geodesically convex submanifold of $\tilde{L}$, then the second fundamental form for F vanishes identically, and it follows that there is no difference between parallel transport (of elements of TF along paths in F) using the ambient metric on $\tilde{L}$ vs. using the inherited metric on F. Consequently, the statement of the following lemma is unambiguous.

Lemma 2.6. *If $v \in S\tilde{L}$ and if $F := \mathrm{FtPar}_{\tilde{L}}(v)$ is a $\mathrm{rk}(v)$-submanifold of $\tilde{L}$, then there is a parallel vectorfield $\mathcal{V}$ on F such that $\mathcal{V}_{\pi(v)} = v$.*

Proof. Let $q := \gamma_v(\infty)$. For each $f \in F$, let $\mathcal{V}_f := \dot{\gamma}_{fq}(0)$; note that $\mathcal{V}_f$ is parallel to v. Then $\mathcal{V}_f$ is parallel to $\mathcal{V}_{f'}$, for all $f, f' \in F$. By the Flat Strip Lemma [10, Proposition 5.1, p. 66], $\mathcal{V}$ is parallel. □

Definition 2.7. If $l \in \tilde{L}$ and $v \in S_l\tilde{L}$, then we define $\mathrm{PJF}_0(v) \subseteq T_l\tilde{L}$ to be the space of initial vectors to parallel Jacobi fields along γ_v. The *rank* of a vector $v \in S\tilde{L}$ is $\mathrm{rk}(v) := \dim(\mathrm{PJF}_0(v))$.

Lemma 2.8. *Let $l \in \tilde{L}$, $v_1, v_2, \ldots \in S_l\tilde{L}$ and $w_1, w_2, \ldots \in S_l\tilde{L}$. Suppose that $v_i \to v$ and that $w_i \to w$ in $S_l\tilde{L}$, as $i \to \infty$. For all $i = 1, 2, \ldots$, assume that $w_i \in \mathrm{PJF}_0(v_i)$. Then $w \in \mathrm{PJF}_0(v)$.*

Proof. For each $i = 1, 2, \ldots$, let J_i denote the extension of w_i to a parallel Jacobi field along γ_{v_i}. Let J denote the extension of w to a parallel vectorfield along γ_v.

Each J_i satisfies the Jacobi equation along γ_{v_i}. In $C^\infty(\mathbb{R}, TL)$, as $i \to \infty$, $J_i \to J$ and $\dot{\gamma}_{v_i} \to \dot{\gamma}_v$. It follows that J is satisfies the Jacobi equation along γ_v. But J is parallel, so the initial vector w of J must lie in $\mathrm{PJF}_0(v)$, as desired. □

Definition 2.9. Let $v_0 \in S\tilde{L}$. We say that v_0 is *regular* if there is a neighborhood U of v_0 in $S\tilde{L}$ such that $\mathrm{rk}(v) = \mathrm{rk}(v_0)$, for all $v \in U$.

Lemma 2.10. *Assume $v_0 \in S\tilde{L}$ is regular. Then there exist Euclidean spaces A and Z, there exists a map $F : A \times Z \to \tilde{L}$ and there exists a neighborhood N of v_0 in $S\tilde{L}$ such that*
(1) $\dim(A) = \mathrm{rk}(v_0)$;
(2) *for every $z \in Z$, $a \mapsto F(a,z) : A \to \tilde{L}$ is an immersion; and*
(3) *for every $v \in N$, there exists $z \in Z$ such that $\pi(v) \in F(A \times \{z\}) \subseteq \mathrm{FtPar}_{\tilde{L}}(v)$.*

Proof. This is a consequence of the involutivity on regular vectors of the distribution $D \subseteq T(S\tilde{L})$ given by parallel Jacobi fields [3, Lemma 2.2, p. 184]. The map F is the composition of a foliation chart (with plaque A and transversal Z) followed by the projection $\pi : S\tilde{L} \to \tilde{L}$. Conclusion (2) of Lemma 2.10 follows from the fact that the tangent spaces to the fibers of π do not contain any nonzero vectors in D. Conclusion (3) follows from the observation [3, p. 184, l.-20 to l.-18] that a D-horizontal path consists of parallel vectors. □

Proposition 2.11. *If $l \in \tilde{L}$ and $v \in S_l\tilde{L}$, then $\mathrm{FtPar}_{\tilde{L}}(v) \subseteq \exp_l(\mathrm{PJF}_0(v))$.*

Proof. This is a consequence of the Flat Strip Lemma [10, Proposition 5.1, p. 66]. □

Lemma 2.12. *Let $v \in S\tilde{L}$ be regular and assume that $\mathrm{FtPar}_{\tilde{L}}(v)$ is a $\mathrm{rk}(v)$-submanifold of $\tilde{L}$. Then there exists a neighborhood N of v in $\mathrm{PJF}_0(v)$ such that: for all $v' \in N$, $\mathrm{PJF}_0(v') = \mathrm{PJF}_0(v)$.*

Proof. Assume the contrary, i.e., that there is a sequence $v_1, v_2, \ldots \in \mathrm{PJF}_0(v)$ such that $v_i \to v$, as $i \to \infty$, and such that $\mathrm{PJF}_0(v_i) \neq \mathrm{PJF}_0(v)$, for all $i = 1, 2, \ldots$.

Let $k := \mathrm{rk}(v)$, let $F := \mathrm{FtPar}_{\tilde{L}}(v)$ and let $l := \pi(v)$. Since v is regular, we may pass to a tail and assume that $\mathrm{rk}(v_i) = k$, for all i. Then, by dimension count, we see that $\mathrm{PJF}_0(v_i) \not\subseteq \mathrm{PJF}_0(v)$. For each i, let J_i denote a parallel Jacobi field on γ_{v_i} such that the initial vector w_i of J_i satisfies $w_i \notin \mathrm{PJF}_0(v)$.

By Proposition 2.11, $F \subseteq \exp_l(\mathrm{PJF}_0(v))$. By Lemma 1.5 (with $A = U = \tilde{L}$), we have $F = \exp_l(\mathrm{PJF}_0(v))$. So $\mathrm{PJF}_0(v) = T_lF$. Therefore, for all i, $v_i \in T_lF$ and $w_i \notin T_lF$.

Fix a positive integer i. Since F is geodesically convex and locally k-smooth in $\tilde{L}$, it follows that $\gamma_{v_i}(\mathbb{R}) \subseteq F$. Further, if $J_i^\perp$ denotes the orthogonal projection of J_i to the normal bundle of F in $\tilde{L}$, then, by [3, Lemma 2.4, p.185], $J_i^\perp$ is again a Jacobi field along γ_{v_i}. Since J_i has constant length, the projection $J_i^\perp$ has bounded length. By [3, Lemma 1.4, p. 181], $J_i^\perp$ is parallel. The initial vector $w_i^\perp$ of $J_i^\perp$ is nonzero and satisfies $w_i^\perp \in \mathrm{PJF}_0(v_i)$.

Let $T_lF^\perp$ denote the normal space to F at l. Then $w_i^\perp \in T_lF^\perp$ and we may, by scalar multiplicaton, assume that $w_i^\perp$ is a unit vector, for all i. Passing to a subseqence, we may assume that $w_1^\perp, w_2^\perp, \ldots$ converges to a unit vector $w^\perp \in T_lF^\perp$. By Lemma 2.8, $w^\perp \in \mathrm{PJF}_0(v)$. But $T_lF = \mathrm{PJF}_0(v)$, so the *unit* vector $w^\perp$ satisfies $w^\perp \in T_lF \cap T_lF^\perp = \{0\}$, a contradiction. □

Lemma 2.13. *Let $v \in S\tilde{L}$ be of rank k. Let $S \subseteq \mathrm{FtPar}_{\tilde{L}}(v)$ and fix $s \in S$. Assume that S is k-smooth on some Euclidean neighborhood of s. Then $\mathrm{FtPar}_{\tilde{L}}(v)$ is k-smooth on some Euclidean neighborhood of s.*

Proof. Let $F := \mathrm{FtPar}_{\tilde{L}}(v)$, let $l := \pi(v)$ and let B be an Euclidean neighborhood of s such that S is k-smooth on B.

By Proposition 2.11, there is a k-dimensional subspace $V \subseteq T_l\tilde{L}$ such that $F \subseteq \exp_l(V)$. Since V is k-smooth on $T_l\tilde{L}$, $\exp_l(V)$ is k-smooth on $\tilde{L}$, so $\exp_l(V)$ is locally k-smooth on B. As $s \in B \cap \exp_l(V)$, it follows that there is an Euclidean neighborhood $N \subseteq B$ such that $\exp_l(V)$ is k-smooth on N.

Now $N \subseteq B$, so S is locally k-smooth on N. By Lemma 1.5, $N \cap S = N \cap \exp_l(V)$. As $S \subseteq F \subseteq \exp_l(V)$, we see that $N \cap F = N \cap \exp_l(V)$, so F is k-smooth on N. □

Lemma 2.14. *Let D be a dense subset of $S\tilde{L}$. For each $v \in D$, assume that $\mathrm{FtPar}_{\tilde{L}}(v)$ is a $\mathrm{rk}(v)$-submanifold of $S\tilde{L}$. Then, for every regular vector $v \in S\tilde{L}$, $\mathrm{FtPar}_{\tilde{L}}(v)$ is a $\mathrm{rk}(v)$-submanifold of $\tilde{L}$.*

Proof. Fix a regular vector $v \in S\tilde{L}$. Let $k := \mathrm{rk}(v)$, let $l := \pi(v)$, let $V := \mathrm{PJF}_0(v)$ and let $F := \mathrm{FtPar}_{\tilde{L}}(v)$. We wish to show that F is a k-submanifold of $\tilde{L}$.

Let $v_1, v_2, \ldots \in D$ be a sequence converging to v. Since v is regular, we may pass to a tail and assume that $\mathrm{rk}(v_i) = k$, for all $i = 1, 2, \ldots$. For each i, let $l_i := \pi(v_i)$, let $F_i := \mathrm{FtPar}_{\tilde{L}}(v_i)$, let $V_i := \mathrm{PJF}_0(v_i)$ and let $\mathcal{B}_i$ be some ordered orthonormal basis of V_i. Passing to a subsequence, we may assume that $\mathcal{B}_1, \mathcal{B}_2, \ldots$ converges to an ordered orthonormal basis for some k-dimensional subspace $V' \subseteq T_l\tilde{L}$. Let $F' := \exp_l(V)$. Since F' is a k-submanifold of $\tilde{L}$, it suffices to show that $F = F'$.

By Proposition 2.11, $F_i \subseteq \exp_{l_i}(V_i)$, for all i. By Lemma 1.5 (with $A = U = \tilde{L}$), $F_i = \exp_{l_i}(V_i)$, for all i.

We claim that $F' \subseteq F$. Fix $f' \in F'$; we will show that $f' \in F$. By Proposition 2.11, $F \subseteq \exp_l(V)$, so $f' = \exp_l(v')$, for some $v' \in V$. Therefore, there exists a sequence $v_1' \in V_1, v_2' \in V_2, \ldots$ such that $\exp_{l_i}(v_i') \to \exp_l(v')$. For all i, let $f_i' := \exp_{l_i}(v_i')$, so that $f_i' \in F_i$ and $f_i' \to f'$. Then, for each i, there is a unit vector footed at f_i' which is parallel to v_i. Passing to a subsequence, these unit vectors converge to a unit vector footed at f'. This unit vector is parallel to v. Therefore $f' \in F$, as desired, proving the claim.

We now know that $F' \subseteq F \subseteq \exp_l(V)$. By Lemma 1.5 (with $A = U = \tilde{L}$), $F' = \exp_l(V)$. We conclude that $F = F'$, as desired. □

We need the Tits metric at infinity which we denote $\mathrm{Td}_{\tilde{L}(\infty)} : \tilde{L}(\infty) \times \tilde{L}(\infty) \to [0, \infty]$. For reference, see [9, Definition 1.5, p. 37] or [4, Section 1.4, p. 33]. Note that none of the statements in [9, Proposition 1.7, p. 37] depend on the duality condition or on assumptions on geometric rank. If $p \in \tilde{L}(\infty)$ and $r > 0$, then we define the *Tits ball about p of radius r* to be $\mathrm{TB}_{\tilde{L}(\infty)}(p, r) := \{q \in \tilde{L}(\infty) \mid \mathrm{Td}_{\tilde{L}(\infty)}(p, q) < r\}$. (The radius of a Tits ball is always assumed to be positive and finite.)

Lemma 2.15. *Let F be a flat in $\tilde{L}$ of dimension $\geqslant 2$. Suppose that $f \in F$ and $p, q \in F(\infty)$. Then $\angle_f(p,q) = \mathrm{Td}_{\tilde{L}(\infty)}(p,q)$.*

Proof. This is a special case of [9, Proposition 1.7(3), p. 37]. $\square$

3. Reverse recurrent vectors

Throughout this section, $\tilde{L}$ denotes a Hadamard manifold. None of the results in Section 3 depend on the duality condition or on the existence of a quotient of $\tilde{L}$ of finite volume.

Definition 3.1. Let $S \subseteq \tilde{L}$, $l \in \tilde{L}$, $\epsilon > 0$, k a nonnegative integer. We say that S is *k-normal ϵ-near l* if there exists a k-dimensional subspace $V \subseteq T_l\tilde{L}$ such that

$$\exp_l(V) \cap B_{\tilde{L}}(l,\epsilon) = S \cap B_{\tilde{L}}(l,\epsilon).$$

Lemma 3.2. *Let $l \in \tilde{L}$, $v \in S_l\tilde{L}$. Define $k := \mathrm{rk}(v)$ and let $\epsilon > 0$. If there exists a subset $S \subseteq \mathrm{FtPar}_{\tilde{L}}(v)$ such that S is locally k-smooth on $B_{\tilde{L}}(l,\epsilon)$, then $\mathrm{FtPar}_{\tilde{L}}(v)$ is k-normal ϵ-near l.*

Proof. Let $F := \mathrm{FtPar}_{\tilde{L}}(v)$, $B := B_{\tilde{L}}(l,\epsilon)$. By Proposition 2.11, there is a k-dimensional subspace $V \subseteq T_l\tilde{L}$ such that $F \subseteq \exp_l(V)$. We wish to show that $B \cap F = B \cap \exp_l(V)$. Since $S \subseteq F \subseteq \exp_l(V)$, it suffices to show that $B \cap S = B \cap \exp_l(V)$. Now V is k-smooth on $\exp_l^{-1}(B)$, so $\exp_l(V)$ is k-smooth on B. So we are done, by Lemma 1.5. $\square$

Fix $\epsilon > 0$ and $v \in S\tilde{L}$. We will say that v is *ϵ-normal* if $\mathrm{FtPar}_{\tilde{L}}(v)$ is $\mathrm{rk}(v)$-normal ϵ-near $\pi(v)$.

Lemma 3.3. *Let $v \in S\tilde{L}$ be ϵ-normal. Then, for all $t \in \mathbb{R}$, $g_t(v)$ is ϵ-normal.*

Proof. Let $k := \mathrm{rk}(v)$.

Fix $t \in \mathbb{R}$ and let $v' := g_t(v)$. Let $l := \pi(v)$, $l' := \pi(v')$. Let $P : T_l\tilde{L} \to T_{l'}\tilde{L}$ denote parallel transport along the geodesic from l to l'. Let $F := \mathrm{FtPar}_{\tilde{L}}(v)$. Then we are assuming that F is k-normal ϵ-near l. We wish to show that F is k-normal ϵ-near l'.

Let $\tilde{F} := \exp_l^{-1}(F)$ and let $\tilde{F}' := \exp_{l'}^{-1}(F)$. Then the Flat Strip Lemma [10, Proposition 5.1, p. 66] implies that $P(\tilde{F}) = \tilde{F}'$.

Let B, B' denote balls of radius ϵ about 0 in $T_l\tilde{L}$, $T_{l'}\tilde{L}$. Then there exists a subspace $V \subseteq T_l\tilde{L}$ such that $\exp_l(V) \cap B_{\tilde{L}}(l,\epsilon) = F \cap B_{\tilde{L}}(l,\epsilon)$. Then $\exp_l(V \cap B) = \exp_l(\tilde{F} \cap B)$, which implies that $V \cap B = \tilde{F} \cap B$, so $P(V) \cap P(B) = P(\tilde{F}) \cap P(B)$.

Let $V' := P(V)$. Then $V' \cap B' = \tilde{F}' \cap B'$, so $\exp_{l'}(V' \cap B') = \exp_{l'}(\tilde{F}' \cap B')$, so $\exp_{l'}(V') \cap B_{\tilde{L}}(l',\epsilon) = F \cap B_{\tilde{L}}(l',\epsilon)$, and we conclude that F is k-normal ϵ-near l'. $\square$

Lemma 3.4. *Let k be a positive integer, let $\epsilon > 0$ and let $l \in \tilde{L}$. Let $w_1, w_2, \ldots \in S_l\tilde{L}$ be a sequence of ϵ-normal vectors converging to $w \in S_l\tilde{L}$. Then there exists a subset $S \subseteq \mathrm{FtPar}_{\tilde{L}}(w)$ such that S is k-normal ϵ-near l. In particular, S is k-smooth on $B_{\tilde{L}}(l, \epsilon)$.*

Proof. For every $n = 1, 2, \ldots$, choose a k-dimensional subspace $V_n \subseteq T_l\tilde{L}$ such that $\exp_l(V_n) \cap B_{\tilde{L}}(l, \epsilon) = \mathrm{FtPar}_{\tilde{L}}(w_n) \cap B_{\tilde{L}}(l, \epsilon)$. Passing to a subsequence, we may assume V_n converges to some k-dimensional subspace $V \subseteq T_l\tilde{L}$. Let $S := \exp_l(V) \cap B_{\tilde{L}}(l, \epsilon)$. □

For every $v \in S\tilde{L}$, $\epsilon > 0$, let $\mathrm{Var}(v, \epsilon)$ denote the set of vectors $w \in S\tilde{L}$ such that $\pi(v) = \pi(w)$ and such that the angle between v and w is $< \epsilon$.

We will say that v is *ϵ-variable* if every vector in $\mathrm{Var}(v, \epsilon)$ is ϵ-normal.

A vector $v \in S\tilde{L}$ is *variable* if there is some $\epsilon > 0$ such that v is ϵ-variable.

Proposition 3.5. *Any regular vector in $S\tilde{L}$ is variable.*

Proof. Let $v_0 \in S\tilde{L}$ be regular. We wish to find an $\epsilon > 0$ such that v_0 is ϵ-variable.

Let $k := \mathrm{rk}(v_0)$. Let $l_0 := \pi(v_0)$. Choose A, Z and F as in Lemma 2.10. Note that $\dim(A) = k$. Choose $(a_0, z_0) \in A \times Z$ such that $F(a_0, z_0) = \pi(v_0)$. By Corollary 1.4, we may choose neighborhoods A_0 of a_0, Z_0 of z_0 and L_0 of l_0 such that, for all $z \in Z_0$, $F_z(A_0) = F(A_0 \times \{z\})$ is locally k-smooth on L_0. Now choose $\epsilon > 0$ such that $\mathrm{Var}(v_0, \epsilon) \subseteq N$ and such that $B_{\tilde{L}}(l_0, \epsilon) \subseteq L_0$.

Let $v \in \mathrm{Var}(v_0, \epsilon)$. We wish to show that v is ϵ-normal, i.e., that $\mathrm{FtPar}_{\tilde{L}}(v)$ is k-normal ϵ-near $\pi(v) = \pi(v_0) = l_0$. By (3) of Lemma 2.10, there exists $z \in Z$ such that

$$l_0 = \pi(v) \in F(A_0 \times \{z\}) \subseteq \mathrm{FtPar}_{\tilde{L}}(v).$$

Now $S := F(A_0 \times \{z\})$ is locally k-smooth on L_0. Then S is locally k-smooth on $B_{\tilde{L}}(l_0, \epsilon)$. By Lemma 3.2, $\mathrm{FtPar}_{\tilde{L}}(v)$ is k-normal ϵ-near l_0. □

We will say that a vector $v \in S\tilde{L}$ is *ϵ-reverse recurrent* if there is a sequence of real numbers $t_1, t_2, \ldots \to \infty$ and such that $\dot{\gamma}_v(-t_i)$ is ϵ-variable in $\tilde{L}$, for all $i = 1, 2, \ldots$.

We will say that a vector $v \in S\tilde{L}$ is *reverse recurrent* if it is ϵ-reverse recurrent for some $\epsilon > 0$.

Proposition 3.6. *Let $v \in S\tilde{L}$ be reverse recurrent. Then $\mathrm{FtPar}_{\tilde{L}}(v)$ is a $\mathrm{rk}(v)$-submanifold of $\tilde{L}$.*

Proof. (Cf. [3, p. 186, l. +5 to p.186, l.-7].)

Let $k := \mathrm{rk}(v)$ and let $F := \mathrm{FtPar}_{\tilde{L}}(v)$. Fix some point $f \in F$. We wish to show that F is k-smooth on some Euclidean neighborhood of f.

Let w denote a vector parallel to v such that $\pi(w) = f$. Choose $\epsilon > 0$ and $t_1, t_2, \ldots \to \infty$ such that $\dot{\gamma}_v(-t_i)$ is ϵ-variable, for all $i = 1, 2, \ldots$.

Fix a positive integer i. Let β_i be a geodesic such that $\beta_i(0) = \gamma_v(-t_i)$ and such that, for some $T_i \geqslant 0$, $\beta_i(T_i) = f$. Let $v_i := \dot{\beta}_i(0)$, $w_i := \dot{\beta}_i(T_i)$. Let ϕ_i denote the angle between $\dot{\gamma}_v(-t_i)$ and v_i. Let ψ_i denote the angle between w_i and w.

By Lemma 2.3, $\phi_i \to 0$ and $\psi_i \to 0$ as $i \to \infty$. In particular, $w_i \to w$ as $i \to \infty$. We may pass to a tail and assume that $\phi_i < \epsilon$, for all $i = 1, 2, \ldots$. Since $\dot{\gamma}_v(-t_i)$ is ϵ-variable, it follows that v_i is ϵ-normal. Then, by Lemma 3.3, w_i is ϵ-normal. Now apply Lemma 3.4 and Lemma 2.13 to conclude that F is k-smooth on some Euclidean neighborhood of f. □

4. Hadamard manifolds with enough flats

Throughout this section, $\tilde{L}$ denotes a Hadamard manifold. None of the results in Section 4 depend on the duality condition or on the existence of a quotient of $\tilde{L}$ of finite volume.

Recall that the *rank* of a vector $v \in S\mathcal{F}$ is the dimension of the space of parallel Jacobi fields along the geodesic γ_v through v.

Definition 4.1. *The geometric rank of $\tilde{L}$ is the minimum of the ranks of the vectors in $S\tilde{L}$. It is denoted* $\mathrm{rk}(\tilde{L})$. *We say that $\tilde{L}$ has enough flats if every vector in $\tilde{L}$ is tangent to some* $\mathrm{rk}(\tilde{L})$*-flat.*

Remark 4.2. Suppose there is a dense subset $D \subseteq S\tilde{L}$ such that every vector in D is tangent to some $\mathrm{rk}(\tilde{L})$-flat. Then $\tilde{L}$ has enough flats.

Proof. Fix $v \in S\tilde{L}$; we wish to find a $\mathrm{rk}(\tilde{L})$-flat F such that $v \in SF$.

Choose $v_1, v_2, \ldots \in D$ converging to v. Let $l := \pi(v)$ and let $l_i := \pi(v_i)$, for $i = 1, 2, \ldots$.

For each $i = 1, 2, \ldots$, choose a $\mathrm{rk}(\tilde{L})$-flat F_i such that $v_i \in SF_i$; then let $\mathcal{B}_i$ denote some ordered orthonormal basis for $T_{l_i}F_i$. After passing to a subsequence, $\mathcal{B}_1, \mathcal{B}_2, \ldots$ converges to an ordered orthonormal basis for a subspace $V \subseteq T_l\tilde{L}$. Let $F := \exp_l(V)$. □

Proposition 4.3. *Let D be a dense subset of $S\tilde{L}$. For each $v \in D$, assume that v is regular and that* $\mathrm{FtPar}_{\tilde{L}}(v)$ *is a* $\mathrm{rk}(v)$*-submanifold of $\tilde{L}$. Then $\tilde{L}$ has enough flats.*

Proof. We repeat here the argument of [3, p. 186, l.-6 to p. 187, l. +8].

Fix $v \in D$. By Remark 4.2, it suffices to show that v is tangent to some $\mathrm{rk}(\tilde{L})$-flat. Let $k := \mathrm{rk}(v)$, let $F := \mathrm{FtPar}_{\tilde{L}}(v)$ and let $l := \pi(v)$. Then $v \in T_lF$ and $\mathrm{rk}(\tilde{L}) \leqslant k = \dim(F)$, so it suffices to show that F is a k-flat.

Since F is geodesically convex and since we are assuming that F is a k-submanifold of $\tilde{L}$, it remains to show that F is isometric to flat Euclidean k-space.

Fix any $f \in F$. The geodesic convexity of F implies that $F = \exp_f(T_fF)$, so, since $\exp_f : T_f\tilde{L} \to \tilde{L}$ is a diffeomorphism, it follows that F is simply connected. Therefore, it suffices to construct a parallel global framing on F.

By Lemma 2.12, there is a neighborhood N of v in $\mathrm{PJF}_0(v)$ such that: for all $v' \in N$, $\mathrm{PJF}_0(v') = \mathrm{PJF}_0(v)$. Let $v_1, \ldots, v_k$ denote a basis for $\mathrm{PJF}_0(v)$ contained in N.

Fix an integer i satisfying $1 \leqslant i \leqslant k$. By Proposition 2.11,

$$\mathrm{FtPar}_{\tilde{L}}(v_i) \subseteq \exp_l(\mathrm{PJF}_0(v_i)) = \exp_l(\mathrm{PJF}_0(v)) = F.$$

By Lemma 2.14, we see that F and $\mathrm{FtPar}_{\tilde{L}}(v_i)$ are k-submanifolds of $\tilde{L}$. By Lemma 1.5 (with $A = U = \tilde{L}$), $F = \mathrm{FtPar}_{\tilde{L}}(v_i)$. By Lemma 2.6, there is a parallel vectorfield $\mathcal{V}^i$ on F such that $\mathcal{V}^i_l = v_i$.

Since $v_1, \ldots, v_k$ is a basis for $T_l F$, we see that $\mathcal{V}^1, \ldots, \mathcal{V}^k$ is a parallel framing for F, as desired. $\square$

Proposition 4.4. *Assume $\tilde{L}$ has enough flats. Let $v \in S\tilde{L}$. Assume $\mathrm{rk}(v) = \mathrm{rk}(\tilde{L})$. Then $\mathrm{FtPar}_{\tilde{L}}(v)$ is a $\mathrm{rk}(\tilde{L})$-flat.*

Proof. Let $k := \mathrm{rk}(\tilde{L})$. Let $l := \pi(v)$. Let F be a k-flat such that $v \in SF$. Then $F \subseteq \mathrm{FtPar}_{\tilde{L}}(v)$. On the other hand, by Proposition 2.11, there exists a k-dimensional subspace $V \subseteq T_l\tilde{L}$ such that $\mathrm{FtPar}_{\tilde{L}}(v) \subseteq \exp_l(V)$. So $F \subseteq \exp_l(V)$. By Lemma 1.5 (with $A = U = \tilde{L}$), $F = \exp_l(V)$, so $F = \mathrm{FtPar}_{\tilde{L}}(v) = \exp_l(V)$. In particular, $\mathrm{FtPar}_{\tilde{L}}(v)$ is a k-flat. $\square$

Lemma 4.5. *Assume that $\tilde{L}$ has enough flats. Let $l \in \tilde{L}$, $v \in S_l\tilde{L}$. Assume that v is regular and that $\mathrm{rk}(v) = \mathrm{rk}(\tilde{L})$. Suppose $v_1, v_2, \ldots \in S_l\tilde{L}$ is a sequence of vectors converging to v. Let $l' \in \tilde{L}$. Then, as $i \to \infty$,*

$$\mathrm{dist}_{\tilde{L}}(l', \mathrm{FtPar}_{\tilde{L}}(v_i)) \to \mathrm{dist}_{\tilde{L}}(l', \mathrm{FtPar}_{\tilde{L}}(v)).$$

Proof. Let $k := \mathrm{rk}(\tilde{L})$.

Since v is regular, we may, by passing to a tail, assume that every v_i has rank k. By Proposition 4.4, $\mathrm{FtPar}_{\tilde{L}}(v)$ is a k-flat and every $\mathrm{FtPar}_{\tilde{L}}(v_i)$ is a k-flat. Then for each $i = 1, 2, \ldots$, there is a k-dimensional subspace $V_i \subseteq T_l\tilde{L}$ such that $\exp_l(V_i) = \mathrm{FtPar}_{\tilde{L}}(v_i)$.

After passing to a subsequence, we may assume that $V_i \to V$, for some k-dimensional subspace $V \subseteq T_l\tilde{L}$. Since every $\exp_l(V_i)$ is a k-flat, it follows that $\exp_l(V)$ is a k-flat. Since $v_i \in V_i$, we have $v \in V$. Thus $\exp_l(V) \subseteq \mathrm{FtPar}_{\tilde{L}}(v)$. By Lemma 1.5 (with $A = U = \tilde{L}$), $\exp_l(V) = \mathrm{FtPar}_{\tilde{L}}(v)$.

We are therefore reduced to showing that $\mathrm{dist}_{\tilde{L}}(l', \exp_l(V_i)) \to \mathrm{dist}_{\tilde{L}}(l', \exp_l(V))$. That is, we must verify that an open metric ball about l' intersects $\exp_l(V)$ iff it intersects all but finitely many of the $\exp_l(V_i)$. By pulling back via the diffeomorphism $\exp_l$ we see that this statement is true, not just for balls about l', but for any precompact open set; it is an exercise in the Grassmannian topology. $\square$

Lemma 4.6. *Assume that $\tilde{L}$ has enough flats. Let $v \in S\tilde{L}$ be regular and assume $\mathrm{rk}(v) = \mathrm{rk}(\tilde{L})$. Let F be a $\mathrm{rk}(\tilde{L})$-flat such that $\gamma_v(-\infty), \gamma_v(\infty) \in F(\infty)$. Then $v \in SF$.*

Proof. Let $k := \mathrm{rk}(\tilde{L})$. Let $p := \gamma_v(-\infty)$ and $q := \gamma_v(\infty)$. Note that $p \neq q$. By Proposition 4.4, $\mathrm{FtPar}_{\tilde{L}}(v)$ is a k-flat. In particular, $\mathrm{FtPar}_{\tilde{L}}(v)$ is k-smooth on $\tilde{L}$.

By Lemma 1.5 (with $A = U = \tilde{L}$), it suffices to show that $F \subseteq \mathrm{FtPar}_{\tilde{L}}(v)$. Let $f \in F$. We wish to show that $f \in \mathrm{FtPar}_{\tilde{L}}(v)$. That is, we wish to show that there is a geodesic $\gamma : \mathbb{R} \to \tilde{L}$ such that $\gamma(0) = f$, $\gamma(-\infty) = p$ and $\gamma(\infty) = q$.

Consider first the special case where $k = 1$. Then there is a geodesic $\gamma : \mathbb{R} \to \tilde{L}$ such that $\gamma(\mathbb{R}) = F$. Reparameterizing, we may assume that $\gamma(0) = f$. Now

$$p, q \in F(\infty) = \{\gamma(-\infty), \gamma(\infty)\},$$

so, after possibly reversing γ, we obtain $\gamma(-\infty) = p$ and $\gamma(\infty) = q$, as desired, concluding the special case $k = 1$.

So we assume that $k \geqslant 2$. It suffices to show that $\angle_f(p, q) = \pi$. Applying Lemma 2.15 to the k-flat $\mathrm{FtPar}_{\tilde{L}}(v)$, we see that $\mathrm{Td}_{\tilde{L}(\infty)}(p, q) = \pi$. Applying Lemma 2.15 to F, we see that $\angle_f(p, q) = \pi$, as desired. $\square$

A vector $v \in S\tilde{L}$ is said to be *ϵ-regular* if

(1) every vector in $\mathrm{Var}(v, \epsilon)$ is regular; and

(2) $\mathrm{rk}(v') = \mathrm{rk}(v)$, for all $v' \in \mathrm{Var}(v, \epsilon)$.

(In fact, (1) implies (2), but we will not need this.) For every positive integer k, the set of regular vectors of rank k is, by definition, an open subset of $S\tilde{L}$. It follows that any regular vector is ϵ-regular for some $\epsilon > 0$.

Lemma 4.7. *Assume $\tilde{L}$ has enough flats. Define $k := \mathrm{rk}(\tilde{L})$. Let $0 < \epsilon < \pi$. Fix $l, l' \in \tilde{L}$. Let $v \in S_l\tilde{L}$ be ϵ-regular of rank k. If there exists a k-flat F such that*

$$l \in F, \qquad \mathrm{Td}_{\tilde{L}(\infty)}(F(\infty), \gamma_v(\infty)) < \epsilon, \qquad \mathrm{dist}_{\tilde{L}}(l', F) \geqslant 1,$$

then there exists a k-flat F' such that

$$l \in F', \qquad \mathrm{Td}_{\tilde{L}(\infty)}(F'(\infty), \gamma_v(\infty)) < \epsilon, \qquad \mathrm{dist}_{\tilde{L}}(l', F') = 1.$$

Proof. Let $p := \gamma_v(\infty)$. Choose $q \in F(\infty)$ such that $T := \mathrm{Td}_{\tilde{L}(\infty)}(p, q) < \epsilon$. It follows, from [9, Proposition 1.7(2), p. 37] and linear reparameterization, that there is a minimizing Tits geodesic $\sigma : [0, T] \to \tilde{L}(\infty)$ such that $\sigma(0) = q$ and $\sigma(T) = p$. Since σ is minimizing, for all $t \in [0, T]$, we have $\mathrm{Td}_{\tilde{L}(\infty)}(p, \sigma(t)) = t \leqslant T < \epsilon$.

For all $t \in [0, T]$, let $w(t) := \dot{\gamma}_{l,\sigma(t)}(0)$. Since σ is $\angle$-continuous, and, in particular, $\angle_l$-continuous, it follows that $w : [0, T] \to S_l\tilde{L}$ is continuous.

Fix $t \in [0, T]$. Then

$$\angle_l(p, \sigma(t)) \leqslant \angle(p, \sigma(t)) \leqslant \mathrm{Td}_{\tilde{L}(\infty)}(p, \sigma(t)) < \epsilon,$$

so $w(t) \in \mathrm{Var}(v, \epsilon)$. Since v is ϵ-regular of rank k, it follows that $w(t)$ is regular of rank k. Then, by Proposition 4.4, $F_t := \mathrm{FtPar}_{\tilde{L}}(w(t))$ is a k-flat. Since $\sigma(t) \in F_t(\infty)$, we see that $\mathrm{Td}_{\tilde{L}(\infty)}(p, F_t(\infty)) \leqslant \mathrm{Td}_{\tilde{L}(\infty)}(p, \sigma(t)) < \epsilon$.

Since $w(0) = \dot\gamma_{lq}(0) \in T_l F$, we have $F \subseteq F_0$. By Lemma 1.5 (with $A = U = \tilde L$), $F = F_0$. By Lemma 4.5, the function

$$d := t \mapsto \mathrm{dist}_{\tilde L}(l', F_t) : [0,T] \to \mathbb{R}$$

is continuous. Since $d(0) \geqslant 1$ and $d(T) = 0$, there must exist some $t_0 \in [0,T]$ such that $d(t_0) = 1$. Let $F' := F_{t_0}$. □

5. Forward recurrent vectors

Throughout this section, $\tilde L$ denotes a Hadamard manifold. None of the results in Section 5 depend on the duality condition or on the existence of a quotient of $\tilde L$ of finite volume.

Let k be a positive integer, let $l \in \tilde L$ and let $A \subseteq \tilde L$. We define the *k-shadow from l of A* to be the set of $p \in \tilde L(\infty)$ such that: any k-flat which contains l and covers p must meet A. Note that what may typically be called the "shadow" is, in this instance, called the "1-shadow".

Lemma 5.1. *Assume that $\tilde L$ has enough flats. Let $v \in S\tilde L$ satisfy* $\mathrm{rk}(v) = \mathrm{rk}(\tilde L)$. *Then there exists $\eta > 0$ such that: if $0 < \epsilon < \eta$, if $t \geqslant 0$ and if $g_{-t}(v)$ is ϵ-regular, then the* $\mathrm{rk}(\tilde L)$*-shadow from $\gamma_v(-t)$ of $B_{\tilde L}(\pi(v), 1)$ contains* $\mathrm{TB}_{\tilde L(\infty)}(\gamma_v(\infty), \epsilon)$.

Proof. This is the argument of [9, Proof of Lemma 4.2, p. 42].

Let $k := \mathrm{rk}(\tilde L)$. Let $l := \pi(v)$ and let $p := \gamma_v(-\infty)$, $q := \gamma_v(\infty)$.

Assume the contrary. Then there exist sequences of real numbers $\epsilon_1, \epsilon_2 \ldots \to 0$ and $t_1, t_2, \ldots \geqslant 0$ such that: for all i, $g_{-t_i}(v)$ is ϵ_i-regular and the k-shadow from $l_i := \gamma_v(-t_i)$ of $B_{\tilde L}(l, 1)$ does not contain $\mathrm{TB}_{\tilde L(\infty)}(q, \epsilon_i)$. Passing to a subsequence, we may assume, for some $x \in \tilde L \cup \tilde L(\infty)$, that $l_i \to x$ in the cone topology on $\tilde L \cup \tilde L(\infty)$.

For each i, choose $q_i \in \mathrm{TB}_{\tilde L(\infty)}(q, \epsilon_i)$ and a k-flat F_i such that $F_i \cap B_{\tilde L}(l, 1) = \emptyset$ and such that $q_i \in F_i(\infty)$. Then $q_i \to q$ in the cone topology on $\tilde L(\infty)$.

Passing to a tail, we may assume that $\epsilon_i < \pi$, for all i. Then by Lemma 4.7, for each i, there exists a flat F_i' such that $l_i \in F_i'$, $\mathrm{dist}_{\tilde L}(l, F_i') = 1$ and $\mathrm{Td}_{\tilde L(\infty)}(q, F_i'(\infty)) < \epsilon_i$. By Lemma 2.5, there is a k-flat F' such that $\mathrm{dist}_{\tilde L}(l, F') = 1$, $x \in F' \cup F'(\infty)$ and $q \in F'(\infty)$.

Since $\mathrm{dist}_{\tilde L}(l, F') = 1$ and $l = \pi(v)$, it follows that $v \notin SF'$.

Consider the special case where $\sup\{t_1, t_2, \ldots\} < \infty$. Then $x \in F'$. But $q \in F'(\infty)$ and F' is geodesically convex, so $\gamma_{xq}(\mathbb{R}) \subseteq F'$. Since $l_i \in \gamma_v(\mathbb{R})$, for all i, we see that $x \in \gamma_v(\mathbb{R})$. But $q = \gamma_v(\infty)$, so $\gamma_v(\mathbb{R}) = \gamma_{xq}(\mathbb{R})$, so $v \in \dot\gamma_v(\mathbb{R}) = \dot\gamma_{xq}(\mathbb{R}) \subseteq SF'$, giving a contradiction, concluding the special case.

So we assume that $t_1, t_2, \ldots$ is unbounded above. Then $x = p$. So $p, q \in F(\infty)$. By Lemma 4.6, $v \in SF'$, again giving a contradiction. □

Definition 5.2. Let $v \in S\tilde L$ and let $\epsilon > 0$. We will say that v is an *ϵ-shadow vector* if:

(1) v is ϵ-regular;

(2) $\mathrm{rk}(v) = \mathrm{rk}(\tilde{L})$; and

(3) for all $t > 0$: if $\dot{\gamma}_v(-t)$ is ϵ-regular, then the $\mathrm{rk}(\tilde{L})$-shadow from $\gamma_v(-t)$ of $B_{\tilde{L}}(\pi(v), 1)$ contains $\mathrm{TB}_{\tilde{L}(\infty)}(\gamma_v(\infty), \epsilon)$.

A vector $v \in S\tilde{L}$ is a *shadow vector* if v is an ϵ-shadow vector, for some $\epsilon > 0$.

Proposition 5.3. *Assume that $\tilde{L}$ has enough flats. Then $S\tilde{L}$ contains a set of positive measure consisting of shadow vectors.*

Proof. Let $k := \mathrm{rk}(\tilde{L})$. By upper semicontinuity of $v \mapsto \mathrm{rk}(v)$, any vector of rank k is regular (see [3, p. 183, l. +8 to l. +9]). The set of regular vectors of rank k is therefore a nonempty open set, and, in particular, is a set of positive measure. Consequently, it suffices to show that any regular vector of rank k is a shadow vector.

Let $v \in S\tilde{L}$ be a regular vector of rank k. Choose η as in Lemma 5.1. Choose $\epsilon > 0$ such that $\epsilon < \eta$ and such that v is ϵ-regular. By Lemma 5.1, v is an ϵ-shadow vector. □

Let $v \in S\tilde{L}$ and let $\epsilon > 0$. We will say that v is *ϵ-forward recurrent* if there exists a sequence $t_1, t_2, \dots \to \infty$ such that, for all $i = 1, 2, \dots$, $g_{t_i}(v)$ is an ϵ-shadow vector.

Let $v \in S\tilde{L}$. We say that v is *forward recurrent* if there exists $\epsilon > 0$ such that v is ϵ-forward recurrent. Since $v \mapsto \mathrm{rk}(v)$ is constant on orbits of the geodesic flow, it follows that if v is forward recurrent, then $\mathrm{rk}(v) = \mathrm{rk}(\tilde{L})$.

Proposition 5.4. *Assume that $\tilde{L}$ has enough flats. Let $\epsilon > 0$. Assume $v \in S\tilde{L}$ is ϵ-forward recurrent. Then $\mathrm{FtPar}_{\tilde{L}}(v)$ is a $\mathrm{rk}(\tilde{L})$-flat which covers $\mathrm{TB}_{\tilde{L}(\infty)}(\gamma_v(\infty), \epsilon)$.*

Proof. Let $k := \mathrm{rk}(\tilde{L})$. Then $\mathrm{rk}(v) = k$. Let $F := \mathrm{FtPar}_{\tilde{L}}(v)$. By Proposition 4.4, F is a k-flat. Let $p := \gamma_v(\infty)$ and fix $q \in \mathrm{TB}_{\tilde{L}(\infty)}(p, \epsilon)$; we wish to show that $q \in F(\infty)$.

Choose $t_1, t_2, \ldots \to \infty$ such that, for all $i = 1, 2, \dots$, the vector $v_i := g_{t_i}(v)$ is an ϵ-shadow vector. Passing to a subsequence, we may assume that $t_i \geqslant t_1$, for all i. Note that $\mathrm{rk}(v_i) = k$, for all i. Let $l_i := \pi(v_i)$, for all i.

Let $w := \dot{\gamma}_{l_1, q}(0)$. Then $w \in \mathrm{Var}(v_1, \epsilon)$ and v_1 is ϵ-regular of rank k, so $\mathrm{rk}(w) = k$. Let $F' := \mathrm{FtPar}_{\tilde{L}}(w)$. By Proposition 4.4, F' is a k-flat.

Note that $l_1 \in F'$ and $q \in F'(\infty)$. Since v_i is a shadow vector and since $t_i - t_1 \geqslant 0$, we have

$$F' \cap B_{\tilde{L}}(l_i, 1) \neq \emptyset,$$

for all i. Choose $l_i' \in F' \cap B_{\tilde{L}}(l_i, 1)$, for all i. Then $l_i' \to p$, so $p \in F'(\infty)$, so $v_1 = \dot{\gamma}_{l_1, p}(0) \in SF'$. Since F' is a flat, we conclude that

$$F' \subseteq \mathrm{FtPar}_{\tilde{L}}(v_1) = \mathrm{FtPar}_{\tilde{L}}(g_{-t_1}(v_1)) = \mathrm{FtPar}_{\tilde{L}}(v) = F.$$

Then $q \in F'(\infty) \subseteq F(\infty)$, as desired. □

6. Tits-visibility, horizon sets and symmetric spaces

Throughout this section, $\tilde{L}$ denotes a Hadamard manifold. None of the results in

Section 6 depend on the duality condition or on the existence of a quotient of $\tilde{L}$ of finite volume.

Lemma 6.1. *A Tits ball covered by a 1-flat consists of a single point.*

Proof. Let F be a 1-flat which covers a Tits ball B. Choose a geodesic $\gamma : \mathbb{R} \to \tilde{L}$ such that $\gamma(\mathbb{R}) = F$. Let $p := \gamma(-\infty)$, $q := \gamma(\infty)$. Then $p \neq q$ and $F(\infty) = \{p, q\}$, so $B \subseteq \{p, q\}$ and it suffices to show that $B \neq \{p, q\}$. Assume for a contradiction that $p, q \in B$.

Since the radius of any Tits ball is finite, $\mathrm{Td}_{\tilde{L}(\infty)}(p, q) < \infty$, so, by [9, Proposition 1.7(2), p. 37], there is a $T > 0$ and a Tits geodesic $\sigma : [0, T] \to \tilde{L}(\infty)$ such that $\sigma(0) = p$ to $\sigma(T) = q$. An open subset of $[0, T]$ containing 0 and 1 cannot be written as a disjoint union of finitely many proper closed subsets of $[0, T]$. In particular, $\sigma^{-1}(B)$ cannot be written as such a disjoint union. If B were finite, then the preimages of points of B would contradict this, so B is infinite. However, $B \subseteq \{p, q\}$, a contradiction. □

Proposition 6.2. *Let k be a positive integer and let B be a Tits ball which is covered by some k-flat. If a k-flat covers the center of B then it covers B.*

Proof. This is the argument of [9, Proof of Lemma 4.3, p. 43]; we repeat it here.

In the special case where $k = 1$, then Lemma 6.1 implies that B consists of a single point. So we may assume $k \geqslant 2$.

Let B have center b and radius r. Let $(S^{k-1}, \angle_0)$ denote the metric space which is the unit sphere in $\mathbb{R}^k$, where the distance between two points is the angle they subtend at the origin. Let B_0 be any ball of radius r in the metric space $(S^{k-1}, \angle_0)$. We leave it as an exercise to show that any distance preserving function from B_0 into B_0 is surjective.

Let F be a k-flat which covers B, let F' be a k-flat which covers b. We wish to show that F' covers B.

Fix $f \in F$ and $f' \in F'$. Let $v \in S_f F$, $v' \in S_{f'} F'$ satisfy $\gamma_v(\infty) = b = \gamma_{v'}(\infty)$. Let $B_f := \mathrm{Var}(v, r)$, $B_{f'} := \mathrm{Var}(v', r)$, Then B_f and $B_{f'}$ are both metric spaces, with the distance functions being given by angle subtended at f and f', respectively. Note that B_f and $B_{f'}$ are isometric to B_0.

By Lemma 2.15, the visual map $w \mapsto \gamma_w(\infty) : B_f \to \tilde{L}(\infty)$ gives an isometry of B_f onto B. Thus we see that the metric spaces B and B_0 are isometric.

By Lemma 2.15, the visual map $w \mapsto \gamma_w(\infty) : B_{f'} \to \tilde{L}(\infty)$ gives a distance preserving function from $B_{f'}$ into B. Since $B_{f'}$ and B are both isometric to B_0, this map must be surjective. Therefore F' covers B. □

If $B \subseteq \tilde{L}(\infty)$, then we say that B is *Tits-visible* if, for all $l \in \tilde{L}$, for all $b, b' \in B$, we have $\angle_l(b, b') = \mathrm{Td}_{\tilde{L}(\infty)}(b, b')$.

Proposition 6.3. *Assume that $\tilde{L}$ has enough flats. Suppose some $\mathrm{rk}(\tilde{L})$-flat in $\tilde{L}$ covers a Tits ball $B \subseteq \tilde{L}(\infty)$. Then B is Tits-visible.*

Proof. This argument appears in the proof of [9, Theorem 4.1, p. 41]; we repeat it.

In the special case where $k = 1$, then Lemma 6.1 implies that B consists of a single point. So we may assume $k \geqslant 2$.

Let b be the center of B. Let $k := \mathrm{rk}(\tilde{L})$. Let F_0 be a k-flat in $\tilde{L}$ which covers B. Let $l \in \tilde{L}$ and let $p, q \in B$. We wish to show that $\angle_l(p,q) = \mathrm{Td}_{\tilde{L}(\infty)}(p,q)$.

Since $\tilde{L}$ has enough flats, there exists a k-flat F whose tangent bundle contains $\dot{\gamma}_{lb}(0)$. By Proposition 6.2, F covers B and, in particular, covers $\{p,q\}$.

Then, by Lemma 2.15, $\angle_l(p,q) = \mathrm{Td}_{\tilde{L}(\infty)}(p,q)$. □

If $\alpha \in [0,\pi]$ and $b \in \tilde{L}(\infty)$, then an *α-antipode of b* is a point $b' \in \tilde{L}(\infty)$ such that $\{b,b'\}$ is Tits-visible and such that $\mathrm{Td}_{\tilde{L}(\infty)}(b,b') = \alpha$. Note that what is typically called an antipode is, in this terminology, a π-antipode. If $\alpha \in [0,\pi]$, then we define $\tilde{L}_\alpha(\infty)$ to be those points in $\tilde{L}(\infty)$ which have an α-antipode. Let

$$\alpha_{\max} := \max\{\alpha \in [0,\pi] \mid \tilde{L}_\alpha(\infty) \neq \emptyset\}.$$

Define $\tilde{L}_{\max}(\infty) := \tilde{L}_{\alpha_{\max}}(\infty)$. Note that $\tilde{L}_{\max}(\infty) \neq \emptyset$.

Proposition 6.4. *If there is a Tits-visible Tits ball at infinity of cardinality $\geqslant 2$, then either $\tilde{L}$ is flat or $\tilde{L}_{\max}(\infty)$ is a proper subset of $\tilde{L}(\infty)$.*

Proof. This is a slight modification of the statement of [9, Lemma 3.2, p. 40]; the proof in [9], in fact, gives exactly this result. □

If $p, q \in \tilde{L}(\infty)$, then we will say that *p sees q* if there is a geodesic γ such that $\gamma(-\infty) = p$ and $\gamma(\infty) = q$. (In particular, $p \neq q$.) A set $B \subseteq \tilde{L}(\infty)$ is an *horizon set for* $\tilde{L}$ if, for any $p \in \tilde{L}(\infty)$, either p sees only points in B or p sees only points in $\tilde{L}(\infty)\backslash B$. That is, either B "fills the horizon" for p or B is invisible to p. Note that B is an horizon set if and only if B is G_e^*-invariant, where G_e^* is defined as in [8, p. 735, l.-5]. We may therefore restate [8, Theorem B, p. 736] as:

Theorem 6.5. *Assume that $\tilde{L}$ is irreducible. Suppose $\tilde{L}(\infty)$ contains a nonempty, proper, closed horizon set. Then $\tilde{L}$ is symmetric of rank $\geqslant 2$.*

Corollary 6.6. *Assume that $\tilde{L}$ is irreducible. If there exists a Tits-visible Tits ball of cardinality $\geqslant 2$ and if $\tilde{L}_{\max}(\infty)$ is an horizon set, then $\tilde{L}$ is symmetric.*

Proof. By Proposition 6.4, either $\tilde{L}$ is flat (hence symmetric) or $\tilde{L}_{\max}(\infty)$ is a nonempty, proper horizon set in $\tilde{L}(\infty)$, in which case we may apply Theorem 6.5. □

7. Splitting the universal cover

Let L be a connected, complete Riemannian manifold of nonpositive sectional curvature. Let $\tilde{L}$ denote the universal cover of L.

Let $\tilde{L} := \tilde{L}_0 \times \tilde{L}_1 \times \cdots \times \tilde{L}_m$ be a decomposition of L such that $\tilde{L}_0$ is flat and $\tilde{L}_1, \ldots, \tilde{L}_m$ are the nonflat irreducible deRham factors. Reordering, we may assume that each of $\tilde{L}_1, \ldots, \tilde{L}_h$ is either symmetric or of geometric rank $= 1$, whereas each of $\tilde{L}_{h+1}, \ldots, \tilde{L}_m$ is both nonsymmetric and of geometric rank $\geqslant 2$. The manifolds $\tilde{L}_1, \ldots, \tilde{L}_m$ will be called the *non-flat factors of* $\tilde{L}$ or the *NF-factors of* $\tilde{L}$. The manifolds $\tilde{L}_{h+1}, \ldots, \tilde{L}_m$ will be called the *nonsymmetric higher rank factors of* $\tilde{L}$ or the *NSHR-factors of* $\tilde{L}$. For every isometry $\phi : \tilde{L} \to \tilde{L}$, there are isometries

$$\begin{aligned}
&\lambda : \tilde{L}_0 \to \tilde{L}_0, \\
&\mu : \tilde{L}_1 \times \cdots \times \tilde{L}_h \to \tilde{L}_1 \times \cdots \times \tilde{L}_h \\
&\nu : \tilde{L}_{h+1} \times \cdots \times \tilde{L}_m \to \tilde{L}_{h+1} \times \cdots \times \tilde{L}_m
\end{aligned}$$

such that $\phi = \lambda \times \mu \times \nu$.

For each $l = (l_0, l_1, \ldots, l_m) \in \tilde{L}$, the decomposition of $\tilde{L}$ induces a decomposition $T_l\tilde{L} = T_{l_0}\tilde{L}_0 \oplus T_{l_1}\tilde{L}_1 \oplus \cdots \oplus T_{l_m}\tilde{L}_m$. For $i = 0, \ldots, m$,

(1) let $D_i(l)$ denote the image in $T_l\tilde{L}$ of $T_{l_i}\tilde{L}_i$,

(2) let $C_i(l)$ denote the image in $S_l\tilde{L}$ of $S_{l_i}\tilde{L}_i$, and

(3) let $D_i := \bigcup_{l\in\tilde{L}} D_i(l)$ and $C_i := \bigcup_{l\in\tilde{L}} C_i(l)$.

Fix an integer i such that $0 \leqslant i \leqslant m$. Define Liouville measure on $S\tilde{L}_i$ normalized so that: if $U \subset \tilde{L}_i$ is open, then the measure of SU is the volume of U. Now $D_i \subseteq T\tilde{L}$ defines an integrable distribution on $\tilde{L}$. Both D_i and C_i are invariant under the geodesic flow. There is a natural bijection

$$C_i \quad \leftrightarrow \quad \tilde{L}_0 \times \cdots \times \tilde{L}_{i-1} \times S\tilde{L}_i \times \tilde{L}_{i+1} \times \cdots \times \tilde{L}_m,$$

and since the right hand side is (via volume forms and normalized Liouville measure) a measure space, we obtain a measure on C_i. If L has finite volume, then $S^{\mathrm{NF}}L$ is a finite measure space. Since the geodesic flow on $S\tilde{L}_i$ is measure preserving, it follows that the geodesic flow on C_i is also measure preserving.

Let $S^{\mathrm{NF}}\tilde{L} := C_1 \cup \cdots \cup C_m$. Give $S^{\mathrm{NF}}\tilde{L}$ the average of the measures on $C_1, \ldots, C_m$. Since $S^{\mathrm{NF}}\tilde{L}$ is invariant under isometries of $\tilde{L}$, it defines a subset $S^{\mathrm{NF}}L \subseteq SL$. Using fundamental domains, we may identify $S^{\mathrm{NF}}L$ as a subset of $S^{\mathrm{NF}}\tilde{L}$, so $S^{\mathrm{NF}}L \subseteq SL$ becomes a measure space. Note that the geodesic flow is measure preserving on $S^{\mathrm{NF}}\tilde{L}$, and hence on $S^{\mathrm{NF}}L$.

We repeat this for the NSHR-factors. Let $S^{\mathrm{NSHR}}\tilde{L} := C_{h+1} \cup \cdots \cup C_m$. Give $S^{\mathrm{NSHR}}\tilde{L}$ the average of the measures on $C_{h+1}, \ldots, C_m$. Since $S^{\mathrm{NSHR}}\tilde{L}$ is invariant under isometries of $\tilde{L}$, it defines a subset $S^{\mathrm{NSHR}}L \subseteq SL$. Using fundamental domains, we may identify $S^{\mathrm{NSHR}}L$ as a subset of $S^{\mathrm{NSHR}}\tilde{L}$, so $S^{\mathrm{NSHR}}L \subseteq SL$ becomes a measure space. Note that the geodesic flow is measure preserving on $S^{\mathrm{NSHR}}\tilde{L}$, and hence on $S^{\mathrm{NSHR}}L$.

For each $0 \leqslant i \leqslant m$, there is a canonical map $\tilde{L}_i(\infty) \to \tilde{L}(\infty)$ whose image we will denote $\tilde{L}_i^*(\infty)$. The image of $(\tilde{L}_i)_{\max}(\infty)$ in $\tilde{L}(\infty)$ will be denoted $(\tilde{L}_i)^*_{\max}(\infty)$. Let

$$\begin{aligned}
\tilde{L}^{\mathrm{NSHR}}(\infty) &:= \tilde{L}^*_{h+1}(\infty) \cup \cdots \cup \tilde{L}^*_m(\infty), \\
\tilde{L}^{\mathrm{NSHR}}_{\max}(\infty) &:= (\tilde{L}_{h+1})^*_{\max}(\infty) \ \cup \cdots \cup \ (\tilde{L}_m)^*_{\max}(\infty).
\end{aligned}$$

Let $\mathrm{Td}^{\mathrm{NSHR}}_{\tilde{L}(\infty)}$ denote the restriction of $\mathrm{Td}_{\tilde{L}(\infty)}$ to $\tilde{L}^{\mathrm{NSHR}}(\infty)$. An *NSHR-Tits ball* is a ball in the metric space $(\tilde{L}^{\mathrm{NSHR}}(\infty), \mathrm{Td}^{\mathrm{NSHR}}_{\tilde{L}(\infty)})$.

A flat F is called an *NSHR-flat* if there exists an integer i such that $h+1 \leqslant i \leqslant m$ and such that F is D_i-horizontal. We say that a a D_i-horizontal flat F is *appropriate dimensional* if $\dim(F) = \mathrm{rk}(\tilde{L}_i)$.

Fix an integer i such that $1 \leqslant i \leqslant m$. The projection $p : \tilde{L} \to \tilde{L}_i$ induces a map $dp|C_i : C_i \to S\tilde{L}_i$. A vector $v \in C_i$ is said to be *NF-variable* if $dp(v) \in S\tilde{L}_i$ is variable. We say that $v \in C_i$ is *NF-reverse recurrent* if $dp(v) \in S\tilde{L}_i$ is reverse recurrent.

Fix an integer i such that $h+1 \leqslant i \leqslant m$. Again, let $p : \tilde{L} \to \tilde{L}_i$ denote projection. We say that $v \in C_i$ is an *NSHR-shadow vector* if $dp(v) \in S\tilde{L}_i$ is a shadow vector. A vector $v \in C_i$ is said to be *NSHR-forward recurrent* if $dp(v) \in S\tilde{L}_i$ is forward recurrent.

This terminology allows us to reformulate in "split form" the essential results from Section 3–6.

Proposition 7.1. *There is a dense open subset of $S^{\mathrm{NF}}\tilde{L}$ consisting of* NF-*variable vectors.*

Proof. This follows from Proposition 3.5 and the open density of regular vectors in any Hadamard manifold, which, in turn, follows from the upper semicontinuity of $v \mapsto \mathrm{rk}(v)$, see [3, p. 183, l. +7]. □

Proposition 7.2. *Assume that $S^{\mathrm{NF}}\tilde{L}$ contains a dense set of* NF-*reverse recurrent vectors. Then every irreducible factor of $\tilde{L}$ has enough flats. In particular, $\tilde{L}$ has enough flats.*

Proof. Any flat factor of $\tilde{L}$ has enough flats, so we reduce to the NF-factors $\tilde{L}_1, \ldots, \tilde{L}_m$.

Fix an integer i such that $1 \leqslant i \leqslant m$. Then $\tilde{L}_i$ has a dense set of reverse recurrent vectors. Let D denote the set of vectors in $S\tilde{L}_i$ which are both regular and reverse recurrent. By open density of regular vectors (which follows from the upper semicontinuity of $v \mapsto \mathrm{rk}(v)$, see [3, p. 183, l. +7]), D is dense in $S\tilde{L}_i$. By Proposition 3.6 and Proposition 4.3, $\tilde{L}_i$ has enough flats. □

Proposition 7.3. *Assume that every* NSHR-*factor of $\tilde{L}$ has enough flats. Assume that $\tilde{L}^{\mathrm{NSHR}}_{\max}(\infty)$ is an horizon set for $\tilde{L}$. Then no appropriate dimensional* NSHR-*flat can cover an* NSHR-*Tits ball.*

Proof. Suppose for a contradiction that an appropriate dimensional NSHR-flat covers an NSHR-Tits ball. Then, for some integer i such that $h+1 \leqslant i \leqslant m$, there is a $\mathrm{rk}(\tilde{L}_i)$-flat in $\tilde{L}_i$ which covers a Tits ball B in $\tilde{L}_i(\infty)$. By Proposition 6.3, B is Tits-visible.

Let $k := \mathrm{rk}(\tilde{L}_i)$. Since $h+1 \leqslant i \leqslant m$, it follows that $k \geqslant 2$.

Let $(S^{k-1}, \angle_0)$ denote the metric space whose underlying set is the unit sphere in $\mathbb{R}^k$ and whose distance is given by angle subtended at the origin. By Lemma 2.15, $(B, \mathrm{Td}_{\tilde{L}(\infty)})$ is isometric to a ball in $(S^{k-1}, \angle_0)$. Since $k \geqslant 2$, we conclude that B is uncountable and, in particular, has cardinality $\geqslant 2$.

Since $\tilde{L}^{\mathrm{NSHR}}_{\max}(\infty)$ is an horizon set for $\tilde{L}$, it follows that $(\tilde{L}_i)_{\max}$ is an horizon set for $\tilde{L}_i$. Then, by Corollary 6.6, we find that $\tilde{L}_i$ is symmetric. Since $h+1 \leqslant i \leqslant m$, this is a contradiction. □

Proposition 7.4. *Assume that $\tilde{L}$ has at least one* NSHR-*factor. Assume that every* NSHR-*factor of $\tilde{L}$ has enough flats. Then $S^{\mathrm{NSHR}}\tilde{L}$ contains a set of positive measure consisting of* NSHR-*shadow vectors.*

Proof. This is a consequence of Proposition 5.3. □

Proposition 7.5. *Assume that $\tilde{L}$ has at least one* NSHR-*factor. Assume that every* NSHR-*factor of $\tilde{L}$ has enough flats. Assume that some vector in $S^{\mathrm{NSHR}}\tilde{L}$ is* NSHR-*forward recurrent. Then some appropriate dimensional* NSHR-*flat covers some* NSHR-*Tits ball.*

Proof. This is a consequence of Proposition 5.4. □

8. Foliations with Hadamard leaves

Let $\mathcal{F}$ be a foliation of a finite measure space M by connected, complete Riemannian manifolds. Assume that $\mathcal{F}$ is measure preserving and that a.e. leaf of $\mathcal{F}$ nonpositive sectional curvature.

The universal cover of a leaf L is denoted $\tilde{L}$.

Let $S^{\mathrm{NF}}\mathcal{F}$ denote the disjoint union of $S^{\mathrm{NF}}L$ over leaves L . Then each $S^{\mathrm{NF}}L$ is a measure space and, by integrating against the transverse measure, $S^{\mathrm{NF}}\mathcal{F}$ becomes a measure space. Since M is a finite measure space, so is $S^{\mathrm{NF}}\mathcal{F}$. The geodesic flow is measure preserving on $S^{\mathrm{NF}}\mathcal{F}$.

Theorem 8.1. *For a.e. leaf L, every irreducible factor of $\tilde{L}$ has enough flats.*

Proof. It follows from Poincaré recurrence and Proposition 7.1 that, for a.e. leaf L, $S^{\mathrm{NF}}\tilde{L}$ contains a dense set of NF-reverse recurrent vectors. By Proposition 7.2, we are done. □

Let $S\mathcal{F}$ denote the disjoint union of SL over leaves L. Then each SL is a measure space, and, by integrating against the transverse measure, $S\mathcal{F}$ becomes a measure space. Since M is a finite measure space, so is $S\mathcal{F}$. The geodesic flow is measure preserving on $S\mathcal{F}$.

Let $\tilde{\mathcal{F}}(\infty)$ denote the set-theoretic disjoint union of $\tilde{L}(\infty)$ over leaves L. A function $f : \tilde{\mathcal{F}}(\infty) \to \mathbb{R}$ is *leafwise deck-invariant* if, for every leaf L, the restriction of f to $\tilde{L}$ is invariant under deck transformations of the covering map $\tilde{L} \to L$. A subset of $\tilde{\mathcal{F}}(\infty)$ is *leafwise deck-invariant* if its characteristic function is leafwise deck-invariant.

A leafwise deck-invariant function $f : \tilde{\mathcal{F}}(\infty) \to \mathbb{R}$ defines a function

$$\mathrm{Vis}(f) : S\mathcal{F} \to \mathbb{R}$$

as follows: for every leaf L, for every $v \in SL$, choose a lift $\tilde{v} \in S\tilde{L}$ of v and define $\mathrm{Vis}(f)(v) := f(\gamma_{\tilde{v}}(\infty))$. We say that f is *visually measurable* if $\mathrm{Vis}(f)$ is a measurable function on $S\mathcal{F}$. A leafwise deck-invariant subset of $\tilde{\mathcal{F}}(\infty)$ is *visually measurable* if its characteristic function is visually measurable.

A subset A of $\tilde{\mathcal{F}}(\infty)$ is said to be *leafwise closed* if, for every leaf L, $A \cap \tilde{L}(\infty)$ is closed in $\tilde{L}(\infty)$. We say that A is *a.e. leafwise horizon* if, for a.e. leaf L, $A \cap \tilde{L}(\infty)$ is an horizon set for $\tilde{L}$. The following is the main ergodic-theoretic tool of this paper.

Theorem 8.2. *Any leafwise deck-invariant, leafwise closed, visually measurable subset of $\tilde{\mathcal{F}}(\infty)$ is a.e. leafwise horizon.*

Proof. Let $A \subseteq \tilde{\mathcal{F}}(\infty)$ be leafwise deck-invariant, leafwise closed and visually measurable. We wish to show that A is leafwise horizon.

We may pass to a measurable $\mathcal{F}$-invariant subset and assume that $A \cap \tilde{L}(\infty) \neq \emptyset$, for every leaf L of $\mathcal{F}$.

Define $\tilde{f} : S\tilde{\mathcal{F}} \to \mathbb{R}$ as follows: for every leaf L, for every $l \in \tilde{L}$, for every $v \in S_l\tilde{L}$, let

$$\tilde{f}(v) := \min\{\angle_l(\gamma_v(\infty), a) \mid a \in A \cap \tilde{L}(\infty)\}.$$

By Lemma 2.2, for every leaf L, $\tilde{f}|S\tilde{L}$ is nondecreasing along the geodesic flow on $S\tilde{L}$.

Since A is leafwise deck-invariant, it follows, for every leaf L, that the function $\tilde{f}|S\tilde{L}$ is invariant under deck transformations of $S\tilde{L} \to SL$ and therefore descends to a function on SL. Thus we obtain a function $f : S\mathcal{F} \to \mathbb{R}$. Since A is visually measurable, it follows that f is measurable. Furthermore, f is nondecreasing along the geodesic flow on $S\mathcal{F}$.

Since the geodesic flow is measure preserving on the finite measure space $S\mathcal{F}$, it follows that f is constant along a.e. orbit of the geodesic flow. Consequently, for a.e. leaf L, $\tilde{f}|S\tilde{L}$ is constant along a.e. orbit of the geodesic flow on $S\tilde{L}$. By continuity, we conclude that, for a.e. leaf L, $\tilde{f}|S\tilde{L}$ is actually constant along *all* orbits of the geodesic flow on $S\tilde{L}$.

Fix any leaf L such that $\tilde{f}|S\tilde{L}$ is constant along orbits in $S\tilde{L}$. We will show that $A \cap \tilde{L}(\infty)$ is horizon. Assume that $a, p, q \in \tilde{L}(\infty)$, that p sees a, that p sees q and that $a \in A$. We wish to show that $q \in A$.

Let γ be a geodesic in $\tilde{L}$ such that $\gamma(-\infty) = p$ and $\gamma(\infty) = q$. Lemma 2.4 implies that $\tilde{f}(\dot{\gamma}(-t)) \to 0$ as $t \to \infty$. Since $\tilde{f}|S\tilde{L}$ is constant along orbits in $S\tilde{L}$, we conclude that $\tilde{f}(\dot{\gamma}(t)) = 0$, for all $t \in \mathbb{R}$. Therefore $q = \gamma(\infty) \in A$. $\square$

Corollary 8.3. *For a.e. leaf L of $\mathcal{F}$, $\tilde{L}^{\mathrm{NSHR}}_{\max}(\infty)$ is an horizon set.*

Let $S^{\mathrm{NSHR}}\mathcal{F}$ denote the disjoint union of $S^{\mathrm{NSHR}}L$ over leaves L . Then each $S^{\mathrm{NSHR}}L$ is a measure space and, by integrating against the transverse measure, $S^{\mathrm{NSHR}}\mathcal{F}$ becomes a measure space. Since M is a finite measure space, so is $S^{\mathrm{NSHR}}\mathcal{F}$. The geodesic flow is measure preserving on $S^{\mathrm{NSHR}}\mathcal{F}$.

Theorem 8.4. *For a.e. leaf L of $\mathcal{F}$, $\tilde{L}$ has no* NSHR*-factors.*

Proof. Assume the contrary. Passing to an $\mathcal{F}$-invariant subset of positive measure, we may assume: for every leaf L of $\mathcal{F}$, $\tilde{L}$ has at least one NSHR-factor.

By Theorem 8.1 and Corollary 8.3, we may assume: for every leaf L,

(1) every irreducible factor of $\tilde{L}$ has enough flats; in particular, every NSHR-factor of $\tilde{L}$ has enough flats; and

(2) $\tilde{L}^{\mathrm{NSHR}}_{\max}(\infty)$ is an horizon set.

By Poincaré recurrence on $S^{\mathrm{NSHR}}\mathcal{F}$ and by Proposition 7.4: for a.e. leaf L, $S^{\mathrm{NSHR}}\tilde{L}$ contains an NSHR-forward recurrent vector. Therefore, by Proposition 7.5, for a.e. leaf L, an appropriate dimensional NSHR-flat covers some NSHR-Tits ball. However, since $\tilde{L}^{\mathrm{NSHR}}_{\max}(\infty)$ is an horizon set, this contradicts Proposition 7.3. $\square$

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